Lesson Five. Calculus

L. Valdes
Department of Mathematics
San Jose State University

Objectives: To be able to

1. find limits of functions
2. find derivatives of functions
3. find partial derivatives and derivatives of higher order
4. integrate functions
5. find the Taylor polynomial approximations of functions

Read Section 4.4 in User Manual

To begin this lesson let us define some functions so they will be at the ready. Say $f(x) = 3x^2 + 19x - 14$, $g(x) = \sin(x) + \cos(3x)$ and $h(x) = \frac{1}{x} + 2$. We will also need some multivariable functions. Type in $fl(x, y) = \frac{3xy}{x^2 + y^2}$ and $gl(x, y) = \sin\left(\frac{x}{y}\right)$.
There is also a package called student that we will be working with so let us open it now. Remember how? *with(student);*

$$f := x \to 3x^2 + 19x - 14$$
$$g := x \to \sin(x) + \cos(3x)$$
$$h := x \to \frac{1}{x} + 2$$
$$fl := (x, y) \to \frac{3xy}{x^2 + y^2}$$
$$gl := (x, y) \to \sin\left(\frac{x}{y}\right) \quad \textbf{(1)}$$

Limits

To find the limit of a function we write limit(function,x=a,direction) where direction is right or left. If direction is omitted, then the limit from both directions is returned. Find $\lim_{x \to -1} f(x)$, $\lim_{x \to \infty} g(x)$, and $\lim_{x \to 0+} h(x)$, $\lim_{x \to 0-} h(x)$.

The command Limit is used when you wish the function to remain inert, that is, to not return a value. Try changing one of the expressions above so that it begins with L. If you wish to use Limit you can follow that command line with *value(%)*; to obtain the value of the limit.

It may be the case that you wish to find the limit of a function of more than one variable. Use the form limit(function,{sequence of points=whatever}); . Find $limit(fl(x, y), \{x = 1, y = 1\})$ and $limit(gl(x, y), \{x = 0, y = 0\})$. What does the output of the second limit mean?

Derivatives

The standard command for finding the derivative of a function is *diff(function,variable or variables)* . Find the derivatives of the functions *f*, *g*, and *h*.

You can find higher derivatives in different ways. The most straight forward is by typing in the variable as a sequence as many times as necessary. That is, if you wish to find the 4th derivative of a function you type *diff(function, x,x,x,x)*. Try it below on the function *g*.

However, a more efficient way may be to use the repetition operator $. Try typing *x$4* and executing it.

Now, enter diff(function, x$4) .

One needs to be a little careful with the independent variable in the function under consideration. Execute the following two statements and be sure you understand the different outputs.

$\frac{d}{dx}\left(a x^2 + b x + c\right)$ and $\frac{d}{dx}\left(a(x) + k f(x)\right)$.

It is often the case that we wish to find the derivative of a function defined implicitly in a relation. The command to use is *implicitdiff* and it takes the form *implicitdiff((equation, dependent variable, independent variable)* .

Find $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ of the relation $x y^2 + x^3 y - 3 x^5$.

How does Maple handle piecewise functions? Check by executing the function below and then by finding the derivative. First plot the function with $-1 \le x \le 6$ to get a better idea of how the function behaves. Finally, use *eval* and the label of the output of the derivative command to find the values of the derivative at $x = -5$, $x = 0$, $x = 2$, and $x = 4$.

```
> gg:=x->piecewise(x<0,x^2+1,x<2,1,3);
```

There is another command that works in much the same way as *diff*. It is the differential operator *D*.

Use it instead of *diff* when you want to output a funcion. In other words, *D* is used for creating functions and *diff* is used to create expressions. To find the derivative of the function *f*, you enter *D(f)*. Try to find the derivatives of *f, g*, and *h* using the differential operator *D*. Then evaluate each of the derivatives at $x = 4$ by using the statement *D(f)(4)*.

The correspondence between the two commands is *D(f)(x) = diff(f(x),x)*. To use the operator *D* to yield higher order derivatives, write *(D@@n)(function)*. In other words, *(D@@n)(function)(x) = diff(function,x$n)*. Find the following:

$D^{(3)}(f)$, $D^{(4)}(g)$, and $D^{(2)}(h)$.

Partial Differentiation

Maple handles partial differentiation in the suspected manner. The independent variable is used as the second argument in the command *diff*. Find

$\frac{\partial}{\partial x} f1$, $\frac{\partial}{\partial y} f1$, $\frac{\partial}{\partial x} g1$, and $\frac{\partial}{\partial y} g1$.

To find higher partial derivatives continue adding arguments. For example, *diff(f,x,y)* = $\frac{\partial^2}{\partial y\, \partial x} f$. Find all the second partials of *f1* and *g1*.

Some people prefer using the differential operator *D* when finding partials. You can be very explicit in which variable you have chosen to differentiate by. When given a function defined in terms of *n* variables and you wish to find the partial with respect to the *ith* variable you type *D[i](f)*. Try finding $D_2(f1)(x, y)$ below and compare it with *diff(f1(x,y),y)*.

Now, if you wish to find $\frac{\partial^2}{\partial y\, \partial x} function(x, y)$ you can use the *diff* command or *D[1,2](f)(x,y)*. And if you wish to find the partial derivative repeatedly with respect to *x*, say n times, you can enter *D[x$n](x,y)*. Use *D* to find all second partials of *f1* and *g1*.

Let us look at one last fun item dealing with differentiation. There is a command, *showtangent*, found in the student package. Execute the following.

As you see, the graph of *f(x)* was drawn along with the tangent line at $x = 1$. Plot *g(x)* and the tangent line at $x = \frac{\pi}{2}$.

Integtration

Integration is straight forward. The format for a definite integral is *int(fnc,x=a..b)* where *fnc* is a fucntion in *x* and *x* ranges from *a* to *b*. Try integrating *f(x)* from -3 to 5 and then find the integral without evaluation. Once again, using *Int* instead of *int* will return the inert function. Try it below. You can use evalf to evaluate Int, but this is not always a wise thing to do in that you may very well get erroneous results.

Integrating piecewise functions is no problem. Integrate *gg* from -5..4.

To find multiple integrals we nest the integrals. To find $\int_{-1}^{2}\int_{0}^{3} f1(x, y)\, dx\, dy$ enter

```
int(int(f1(x,y),x = 0 .. 3),y = -1 .. 2);
```

. Find $\int_{1}^{2}\int_{1}^{2} g1(x, y)\, dx\, dy$ and evaluate.

Taylor Polynomials

To find the Taylor series approximation of a function centered at $x = a$ to the *nth* degree use the command

tayor(function,x=a,n). Find the Taylor polynomial approximation of $r(x) = e^x$ centered at *x=1* to the 8th degree.

The output includes the order of an 8th degree polynomial. This output is not a polynomial and if you wished to plot it, you would have to delete that last expression. To do so, use *convert (expression,polynom).* Then plot e^x and the Taylor polynomial on the same set of axes. Use a range of -1..10.